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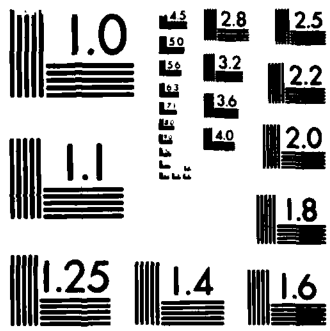

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ON THE ANTI-CYLINDER

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ON THE ANTI-CYLINDER

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ABSTRACT

In 1948 Villarceau discovered the slanting circles on the surface of a horizontal torus T (see [4] and also [1]). Let Γ be the central circle of T, of radius a, which is the locus of the center of a sphere of radius $c(c < a)$, which envelopes the torus T. Let Γ' and Γ'' be the Villarceau circles defined by the intersection $\pi \cap T$ of T with a slanting plane π which is twice tangent to T. It was shown in [3] that T' can be obtained from Γ by a translation by the vector $\overrightarrow{AA'}$ of length c along its diameter AC (see Fig. 1) followed by a rotation by the angle α, where $\sin \alpha = c/a$, around its diameter AC. The anti-cylinder $AC(\Gamma,\Gamma')$ is the figure having Γ as bottom-circle and Γ' as top-circle. From $\Gamma' \subset T$ we see that Γ and Γ' have a 1-parameter family of common normals all having the same length $= c$; these are the generatrices of the anti-cylinder. Our main result is this: On the basis of the remarkable paper [1] it is shown that there is a 1-parameter family of skew rhombuses, all having their sides $= (2a^2 - c^2)^{1/2}$, which zigzag between the two circles Γ and Γ'. Fig. 2 shows the anti-cylinder with 16 generatrices, of which the author made a model $1\frac{1}{2}$ feet high. If made of anodized aluminum tubing and of reasonable height (15 to 20 feet), Fig. 2 would suggest an attractive out-door sculpture, of interest also to non-mathematicians.

AMS (MOS) Subject Classifications: 51-01, 70B15
Key Words: the cylinder, The Villarceau circles of a torus
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SIGNIFICANCE AND EXPLANATION

~~We~~ define a figure which ~~we~~ call an anti-cylinder. Like the cylinder, it has a bottom circle Γ and a topcircle Γ' and all common normals of Γ and Γ' have the same length c. ~~Fig.~~ 2 exhibits a model made by the author of 1 1/2 feet height. Unlike the two circles of a cylinder, the circles Γ and Γ' are linked, like two consecutive elements in a chain. If made of anodized aluminum tubing and of reasonable height (15 to 20 feet) it would make an attractive out-door sculpture, of interest also to non-mathematicians. Also perhaps for children's playgrounds, the children climbing around the figure. Keywords:

ON THE ANTI-CYLINDER

I. J. Schoenberg

1. INTRODUCTION.

The anti-cylinder. Everybody knows what a cylinder is, one of the most important figures in Geometry. The cylinder $C(\Gamma, c)$ is defined by two equal circles Γ and Γ_1 having a common axis whose parallel planes are at the distance c the height of the cylinder. Γ and Γ_1 have a 1-parameter family of common normals, all of length c; these are the generatrices of the cylinder. But what is an anti-cylinder? Except for the name it was introduced by the author in [3]. The easiest way to define it is to start from a torus T. Let $T = T(\Gamma, c)$ denote the torus enveloped by a sphere of radius c whose center describes a horizontal circle Γ of radius a, $a > c$.

The French astronomer Ivon Villarceau discovered in 1848 that a slanting plane π which is tangent to T in two (hyperbolic) points, intersects T in two circles Γ' and Γ'', both congruent to the central circle Γ (See [4] and also [1]). The anti-cylinder $A-C(\Gamma, \Gamma')$ is the figure in R^3 having Γ as bottom-circle and Γ' as top-circle. Because $\Gamma' \in T(\Gamma, c)$ it is clear that Γ and Γ' have a 1-parameter family of common normals $PP'(P \in \Gamma, P' \in \Gamma')$, all of length $PP' = C$, being all radii of the torus. See Fig. 1, and Fig. 2. In analogy with the case of the cylinder, these common normals PP' are the generatrices of the anti-cylinder. The upper part of Fig. 2 shows an anti-cylinder in which we shall establish some new properties of the anti-cylinder.

In [3] it was shown that Γ' can be obtained from Γ by the following two rigid motions:

Figure 1 shows the horizontal circle $\Gamma = ABC$ of center 0 and radius $a = OA = OB = OC$, with OB perpendicular to the diameter AOC. We first translate the circle $\Gamma = ABC$ by the vector $\overrightarrow{AA'}$, of length c, to its new position $\Gamma_1 = A'FC'$ having the center O'. Let $D = \Gamma \quad O'F$ hence

$$b := (a^2 - c^2)^{1/2} = O'D .$$

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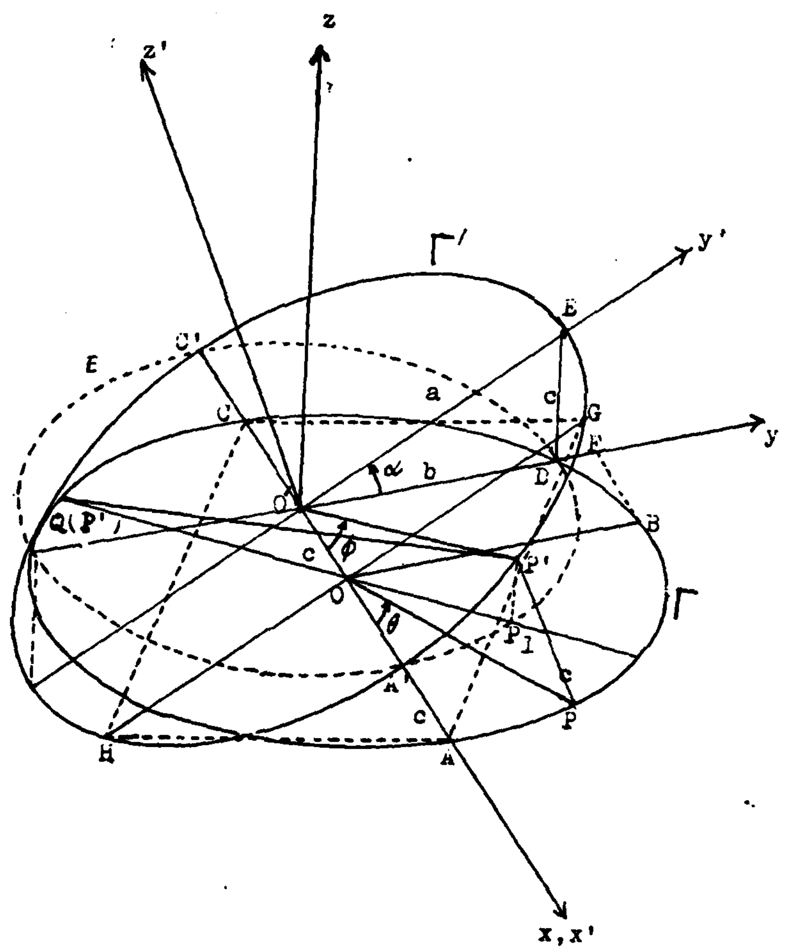
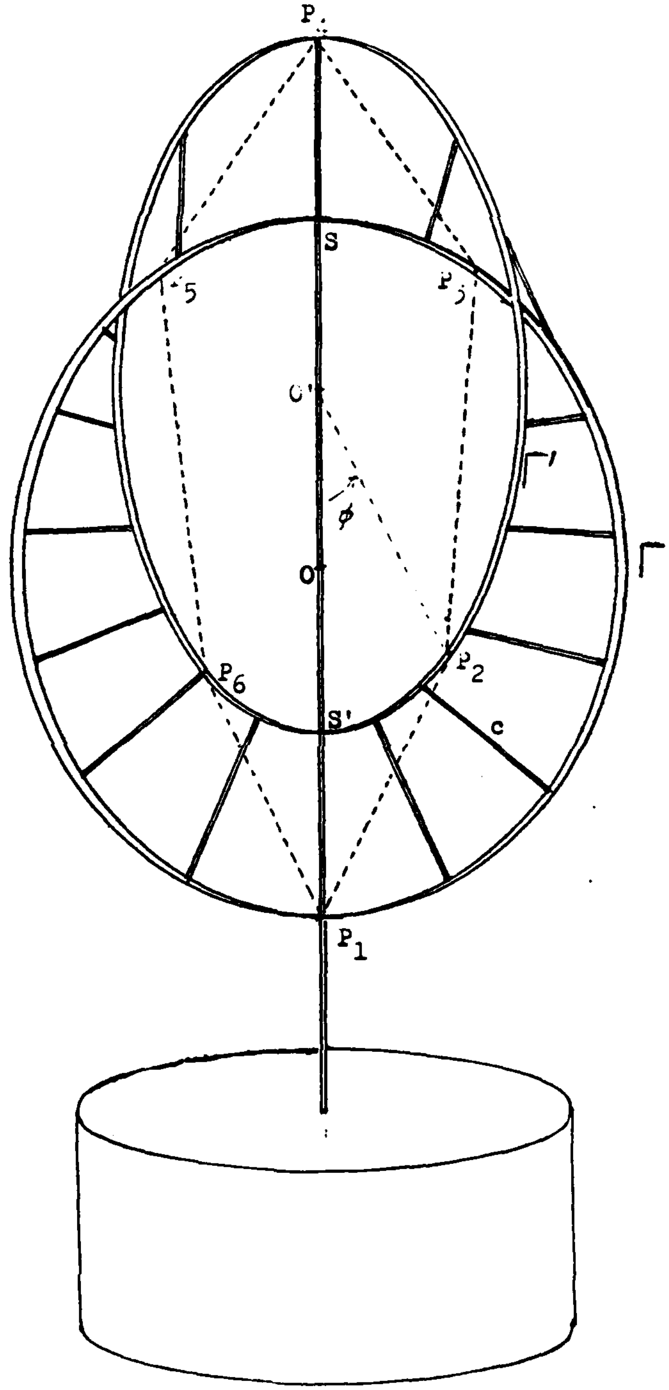


Fig.1



A cylinder carrying an anti-cylinder

Fig. 2

At D we erect the vertical segment $DE = c$. From the congruence of the triangles $O'DE$ and $DO'O$ we see that $OE' = a$. It follows that the angle $\alpha = \sphericalangle DO'E$ is defined by the equations

$$\cos\alpha = \frac{b}{a}, \quad \sin\alpha = \frac{c}{a}. \tag{1.1}$$

Our second rigid motion is this: We turn the circle $\Gamma_1 = A'FC'$, about its diameter $A'C'$ by the angle $\alpha = \sphericalangle FO'E$ obtaining the Villarceau circle

$$\Gamma' = A'EC'. \tag{1.2}$$

The segment $DE = c$ is seen to be a common normal of Γ and Γ'. Therefore Γ and Γ' have three common normals AA', CC', and DE, all having the length c. It seems remarkable that Γ and Γ' have a 1-parameter family of common normals PP', all of length $= c$ (See[3]).

The two rigid motions which move Γ into Γ', can be reversed: First we translate Γ' along CA by the vector $\overrightarrow{A'A}$, and follow this by a rotation around CA by the angle $-\alpha$, obtaining the circle Γ. This shows that Γ is a Villarceau circle of the torus $T(\Gamma',c)$: The relationship between the circles Γ and Γ' is therefore a symmetrical one.

Fig. 2 shows the anti-cylinder $A - C(\Gamma,\Gamma')$ with the line $AA'CC'$ in a vertical position, exhibiting 16 generatrices. These are all of the same length $= c$ and are all common normals of both circles. Fig. 2 is to suggest that the anti-cylinder would furnish an attractive out-door sculpture, of from 10 to 20 feet in height, in which all lines drawn are made of anodised aluminum tubing. Also perhaps for children's playgrounds, the children climbing around the figure. The author made a model, shown in Fig. 2, of 1 1/2 feet in height.

We wish to derive some new properties of the anti-cylinder from a general recent result from [1].

The theorem of W.L. Black, H.C. Howland and B. Howland. We need a remarkable theorem from [1] that recalls the classical theorems of Poncelet and Steiner from the golden age of Geometry. This result can be described as follows.

Let Γ and Γ' be two circles in the 3-dimensional space R^3 , not necessarily of equal radii, and satisfying the

Assumption (1). There exists a distance d such that each point of either circle is at the distance d from exactly two points of the other circle.

Furthermore, the authors assume that the distance d , enjoying the Assumption (1), also has the following property: There is an equilateral closed polygon of $2n$ sides

$$\Pi_{2n} = P_1 P_2 \ldots P_{2n}, \quad (n > 2) \quad , \tag{1.3}$$

having all its sides $= d$ such that

$$P_{2k-1} \in \Gamma \ , \ P_{2k} \in \Gamma' \ , \ (k = 1,2,\ldots,n) \quad . \tag{1.4}$$

We also assume that the polygon $\{P_{2k-1}\}$ is convex (inscribed in Γ). and similarly $\{P_{2k}\}$ is convex (inscribed in Γ'). We may roughly describe the situation by saying that the polygon (1.3) zigzags between the circles Γ and Γ' .

There main result is

Theorem 1 (Black Howland and Howland). There is a 1- parameter family of polygons (1.3) enjoying all properties mentioned, such that P_1 can assume any position on the circle Γ .

We may again roughly describe this theorem by saying that the polygon Π_{2n} can be turned around between the two circles.

There is one case when Theorem 1 is trivial: Γ and Γ' are the top and bottom circles of a circular cylinder, the turning property being obvious due to the rotational invariance of the cylinder. The zigzaging polygon Π_{2n} in this case reminds one of the decorations often seen on the big drums of marching bands.

2. Explicit applications of Theorem 1 to the anti-cylinder for $n = 2$ and $n = 3$. In [1] the authors do not describe the value of the distance d which is the common length of the sides of the polygon (1.3). Our main results are the explicit values of d for $n = 2$ and also for $n = 3$, when Γ and Γ' define the $A - C(\Gamma,\Gamma')$. For $n = 2$ (1.3) may be called a skew rhombus, while for $n = 3$ (1.3) is a closed equilateral hexagon. Our first result is

Theorem 2. The side d of the skew rhombus.

(2.1) $$\Pi_4 = P_1P_2P_3P_4$$

which can "turn" between the circles Γ and Γ' of an anti-cylinder $A - C(\Gamma,\Gamma')$, of "radius" a and "height" c , is given by the equation

(2.2) $$d^2 = a^2 + b^2 = 2a^2 - c^2 \quad .$$

For $n = 3$ we have

Theorem 3. The side d of the closed equilateral hexagon

(2.3) $$\Pi_6 = P_1P_2P_3P_4P_5P_6$$

which can "turn" between the circles Γ and Γ' of the anti-cylinder $A - C(\Gamma,\Gamma')$, of "radius" a and "height" c is given by the equation

(2.4) $$d^2 = 2a^2 + c(2a + c) + \frac{a(a + c)}{a + b}\{a - c - \sqrt{(a - c)^2 + 4(a + b)(b + c)}\} \quad ,$$

where $b = \sqrt{a^2 - c^2}$.

3. A proof of Theorem 2. Deriving Theorem 2 from Theorem 1 requires to establish the following two statements:

(i) We are to exhibit a skew rhombus (2.1), having the side $d = (a^2 + b^2)^{1/2}$, such that

(3.1) P_1 and $P_3 \in \Gamma$, P_2 and $P_4 \in \Gamma'$.

(ii) We are to show that the distance d satisfies the Assumption (1) of Theorem 1.

The requirement (i) is readily met: Returning to Fig. 1 we erect in the plane of Γ' the line GOH normal to $A'C'$ and mark its intersections G and H with Γ' . From $OO' = c$ and $O'H = O'G = a$ we conclude that $OG = OH = b$ and therefore that

$$AG = GC = CH = HA = \sqrt{a^2 + b^2} = d \quad .$$

The rhombus (2.1), with

(3.2) $$P_1 = A\ ,\ P_2 = G\ ,\ P_3 = C\ ,\ P_4 = H\quad ,$$

which happens to be a plane rhombus, satisfies the requirement (3.1) of (i).

Concerning the requirement (ii) we observe the following: The symmetry between Γ and Γ' shows that it suffices to consider only the circle Γ' and to show that if we

pick P' on Γ' , then there are exactly two points P on Γ such that $P'P = d$.
This requires to prove

Lemma 1. Given the point P' on Γ' , then

$$(3.3)\qquad \min_{P \in \Gamma} P'P < d < \max_{P \in \Gamma} P'P .$$

Proof of Lemma 1. From [3] we know that

$$\min_{P \in \Gamma} P'P = c \quad \text{for every} \quad P' \in \Gamma' .$$

and so the first inequality (3.3) amounts to $c < d = \sqrt{a^2 + b^2}$, which is obvious, since $c < a$.

There remains to prove the second inequality (3.3) for every P' on Γ' . We begin by projecting orthogonaly Γ' onto the plane of Γ obtaining the ellipse E . But then P' projects into a point P_1 of E . We now join P_1 to O , and extend P_1O beyond O until it intersects Γ in the point $Q = Q(P')$.

Observe now that the segments $P'P$, for all $P \in \Gamma$, are the generatrices of a second degree cone having the vertex P' and the base Γ . But then obviously

$$(3.4)\qquad \max_{P \in \Gamma} P'P = P'Q(P') .$$

The second inequality (3.3) will therefore be established as soon as we show that

$$(3.5)\qquad \min_{P' \in \Gamma'} P'Q(P') \ \text{is reached for} \ P' = A' \ \text{with the value} \ A'Q(A') = 2a - C$$

and that

$$(3.6)\qquad d = \sqrt{a^2 + b^2} < 2a - C .$$

The last inequality is easily shown: We are to show that $a^2 + b^2 < 4a^2 - 4ac + c^2$. Since $b^2 = a^2 - c^2$, this amounts to $2a^2 - c^2 < 4a^2 - 4ac + c^2$. Equivalently $0 < 2a^2 - 4ac + 2c^2$ which is $0 < (a-c)^2$. This follows from $a > c$.

To prove (3.5) we need the coordinates of P' in the $O'xyz$ system of Fig. 1. In the $O'x'y'z'$ system, with $\phi = \angle A'O'P'$, we have the coordinates

$$P': \quad x' = a\cos\phi \,,\; y' = a\sin\phi \,,\; z' = 0 \;.$$

In the $O'xyz$ system we therefore have for P' the coordinate $x = a\cos\phi$, $y = a\sin\phi\cos\alpha = a\sin\phi(b/a) = b\sin\phi$, and $z = a\sin\phi\sin\alpha = a\sin\phi(c/a) = c\sin\phi$.

We have just found that

$$P': \quad x = a\cos\phi \,,\; y = b\sin\phi \,,\; z = c\sin\phi \;. \tag{3.7}$$

For the projection of P' on the plane of Γ we therefore have

$$P_1 = (a\cos\phi \,,\; b\sin\phi)$$

and therefore the components of the vector $\overrightarrow{OP_1}$ are $\overrightarrow{OP_1} = (a\cos\phi - c \,,\; b\sin\phi)$.

But then

$$OP_1 = \sqrt{E} \;, \tag{3.8}$$

where we write

$$E = (a\cos\phi - c)^2 + b^2\sin^2\phi = c^2 - 2ac\cos\phi + a^2\cos^2\phi + b^2\sin^2\phi \;. \tag{3.9}$$

To prove (3.5) we study the function

$$F(\phi) := (P'Q(P'))^2 \quad \text{on the circle } \Gamma' \;. \tag{3.10}$$

From (3.8) and (3.7) we have $Q(P')P_1 = a + \sqrt{E}$, $P_1P' = c\sin\phi$, and so

$$F(\phi) = (a + \sqrt{E}\,)^2 + c^2\sin^2\phi = a^2 + E + 2a\sqrt{E} + c^2\sin^2\phi$$

or

$$F(\phi) = a^2 + (a\cos\phi - c)^2 + b^2\sin^2\phi + c^2\sin^2\phi + 2a\sqrt{E} \;.$$

Since $a^2 = b^2 + c^2$ we have the final result

$$F(\phi) = a^2 + (a\cos\phi - c)^2 + a^2\sin^2\phi + 2a\sqrt{E} \tag{3.11}$$

Now we wish to show that

(3.12) $F(\phi)$ <u>increases in the interval</u> $0 < \phi < \pi$ <u>and decreases in</u> $\pi < \phi < 2\pi$

Indeed, from (3.11) we find

$$F'(\phi) = 2(a\cos\phi - c)(-a\sin\phi) + 2a^2\sin\phi\cos\phi + \frac{a}{\sqrt{E}}\frac{dE}{d\phi} \;,$$

hence

$$F'(\phi) = 2ac\sin\phi + \frac{a}{\sqrt{E}}\frac{dE}{d\phi} \;. \tag{3.13}$$

However, from (3.9) we find

$$\frac{dE}{d\phi} = 2ac\sin\phi - 2a^2\cos\phi\sin\phi + 2b^2\sin\phi\cos\phi = 2ac\sin\phi - 2c^2\sin\phi\cos\phi \;,$$

or

(3.14) $$\frac{1}{2}\frac{dE}{d\phi} = (a - c \cos\phi)c \sin\phi \quad .$$

Since $E > 0$ fc‹ all ϕ , and $a > c$, (3.13) and (3.14) establish the statement (3.12). This proves Lemma 1 and thereby Theorem 2.

4. A proof of Theorem 3. As in the case of Theorem 2 a proof of Theorem 3 requires to establish the two statements:

(i') We are to exhibit a hexagon (2.3) having all its sides equal to the distance d as given by (2.4), such that

(4.1) $$P_1, P_3, P_5 \text{ are on } \Gamma \text{ and } P_2, P_4, P_6 \text{ on } \Gamma' \quad .$$

(ii') We are to show that this distance d, as given by (2.4), satisfies the Assumption (1) of Theorem 1.

In order to meet the requirement (i') we begin by describing the hexagon Π_6 shown with dotted sides in Fig. 2. We choose $P_1 = A$ in Fig. 1; P_1 is therefore the lowest point of Γ in Fig. 2. From the symmetry of the $A - C(\Gamma, \Gamma')$ with respect to the vertical line which is the common diameter of the two circles, it should be clear that it *suffices to construct one half of the hexagon*, namely $P_1P_2P_3P_4$, with P_4 in the highest point of Γ' . Also that P_2 and P_3 are the correct points, provided that they meet the following requirement:

(4.2) $$\text{The arc } \overset{\frown}{S'P_2} \text{ is equal to the arc } \overset{\frown}{SP_3} \quad ,$$

and

(4.3) $$P_1P_2 = P_2P_3 \quad .$$

For indeed (4.2) implies that the triangles $P_1S'P_2$ and P_4SP_3 are congruent and in particular that

(4.4) $$P_1P_2 = P_3P_4 \quad ,$$

while (4.3) and (4.4) imply that $P_1P_2 = P_2P_3 = P_3P_4$. If we now denote by P_5 and P_6 the points symmetric to P_3 and P_2 , respectively, then $\Pi_6 = P_1P_2P_3P_4P_5P_6$ is the hexagon with the zigzag property. Let us show that the value d given by (2.4), is obtained by enforcing the two conditions (4.2) and (4.3).

In the coordinate system $O'xyz$ of Fig. 1, but referring to Fig. 2, we have

(4.5) $$P_1 = A = (a + c, o, o) \quad .$$

From (3.7) we gather that the coordinates of P_2 are

(4.6) $$P_2 = (a \cos\phi, b \sin\phi, c \sin\phi) \text{ , where } \phi = \angle P_1 O' P_2 \text{ (Fig. 2).}$$

Also that for P_3 we have

(4.7) $$P_3 = (c + a \cos\theta, a \sin\theta, 0) \text{ , where } \theta = \angle P_1 O P_3 \quad .$$

However, the equality (4.2) of the arcs $\widehat{S'P_2}$ and $\widehat{SP_3}$ implies that

$$\theta = \pi - \phi$$

and so $\cos\theta = -\cos\phi$, $\sin\theta = \sin\phi$. Therefore (4.7) becomes

(4.8) $$P_3 = (c - a \cos\phi, a \sin\phi, 0) \quad .$$

From (4.5), (4.6), and (4.8) we obtain

$$(P_1P_2)^2 = (a + c - a \cos\phi)^2 + b^2 \sin^2\phi + c^2 \sin^2\phi$$
$$= a^2 + a^2\cos^2\phi + c^2 - 2a^2\cos\phi + 2ac - 2ac \cos\phi + a^2\sin^2\phi \quad ,$$

because $b^2 + c^2 = a^2$, and therefore

(4.9) $$(P_1P_2)^2 = 2a^2 - 2a(a + c)\cos\phi + c(2a + c) \quad .$$

On the other hand (4.6) and (4.8) show that

$$(P_2P_3)^2 = (c - 2a \cos\phi)^2 + (a \sin\phi - b \sin\phi)^2 + c^2\sin^2\phi$$

or

(4.10) $$(P_2P_3)^2 = c^2 - 4ac \cos\phi + 4a^2\cos^2\phi + (a - b)^2\sin^2\phi + c^2\sin^2\phi \quad .$$

In terms of the angle ϕ the desired equation (4.3) is found to be

(4.11) $$(a + b)\cos^2\phi + (a - c)\cos\phi - (b + c) = 0 \quad .$$

From Fig. 2 we see that $\cos\phi$ must be positive and this quadratic equation therefore gives the value

(4.12) $$\cos\phi = \frac{-(a - c) + \sqrt{(a - c)^2 + 4(a + b)(b + c)}}{2(a + b)} \quad .$$

Substituting this into (4.9) we find $d = P_1P_2$ to be indeed given by the equation (2.4) of Theorem 3. This establishes Theorem 3.

Finally, the requirement (ii') is readily verified. We are to show that the new value $d = d_6$ for the hexagon also satisfies the inequalities (3.3) of Lemma 1. But this is clear, for $d = d_6$ is in the first place greater than the first term of (3.3) which is $= c$, and also less than the second term which is $d = d_4$. This completes our proof of Theorem 3.

The following concluding remarks seem intuitively obvious in view of [3], and would not be hard to establish: 1. That there is, for every $n > 3$ a distance $d = d_n$, which is the side of a polygon (1.3) that zigzags between the circles of an $A - C(\Gamma, \Gamma')$ of height c. 2. That the sequence $\{d_n\}$ is strictly decreasing, 3. That

$$\lim_{n\to\infty} d_n = c \quad .$$

For the model represented by Fig. 2 we have in centimeters

$$a = 10,\ c = 6,\ b = \sqrt{164} = 12.80625 \quad .$$

From (2.4) we find for $d = P_1P_2$ the value $d = 9.59900$ cms. From (4.12) we obtain $\cos\phi = .82456$ whence $\phi = 34^\circ 28' = \angle P_1 0' P_2$, an angle shown in Fig. 2.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In 1948 Villarceau discovered the slanting circles on the surface of a horizontal torus T (see [4] and also [1]). Let Γ be the central circle of T, of radius a, which is the locus of the center of a sphere of radius c $(c < a)$, which envelopes the torus T. Let Γ' and Γ'' be the Villarceau circles defined by the intersection $\pi \cap T$ of T with a slanting plane π which is twice tangent to T. It was shown in [3] that T' can be obtained from Γ by a translation by the vector $\overrightarrow{AA'}$ of length c along its diameter AC (see Fig. 1) followed by a rotation by the angle α, where $\sin \alpha = c/a$,

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20. ABSTRACT - cont'd.

around its diameter AC. The anti-cylinder $AC(\Gamma,\Gamma')$ is the figure having Γ as bottom-circle and Γ' as top-circle. From $\Gamma' \subset T$ we see that Γ and Γ' have a 1-parameter family of common normals all having the same length = c; these are the generatrices of the anti-cylinder. Our main result is this: On the basis of the remarkable paper [1] it is shown that there is a 1-parameter family of skew rhombuses, all having their sides $= (2a^2 - c^2)^{1/2}$, which zigzag between the two circles Γ and Γ'. Fig. 2 shows the anti-cylinder with 16 generatrices, of which the author made a model 1 1/2 feet high. If made of anodized aluminum tubing and of reasonable height (15 to 20 feet), Fig. 2 would suggest an attractive out-door sculpture, of interest also to non-mathematicians.

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